Uncertainty Quantification for Synthetic Controls

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Jun 3, 2022

DataX Synthetic Control Methods Workshop

Introduction

This presentation: a mixture of two papers

- "Prediction Intervals for Synthetic Control Methods," with Matias Cattaneo and Rocío Titiunik. *Journal of the American Statistical Association*, 116(536): 1865-1880, Dec 2021.
- "Uncertainty Quantification in Synthetic Controls with Staggered Treatment Adoption," with Matias Cattaneo, Filippo Palomba, Rocío Titiunik.

I will mostly focus on the canonical case, but the idea can be generalized

Introduction

Synthetic control (SC) method

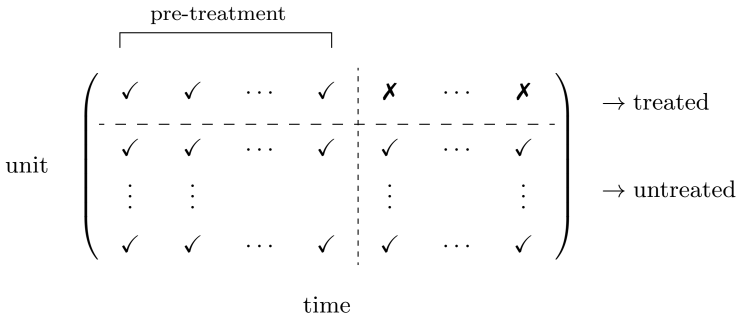
- Introduced by Abadie and Gardeazabal (2003)
- Causal effect of a treatment on a single or a few units
- Popular in comparative case studies
  - German reunification, California Tobacco Control, Compulsory Voting, etc
- Key idea
  - Predict $Y(0)$ of the treated by a linear combination of control units

We focus on uncertainty quantification of SC

- Prediction Intervals for treatment effect on the treated
- Two sources of uncertainty
- Non-asymptotic coverage guarantees
- Practical guidance
- Software package `scpi`

Data Structure: Canonical Case

Treatment: $D_{it} = 0, 1$ Outcome: $Y_{it} = Y_{it}(1)D_{it} + Y_{it}(0)(1 - D_{it})$

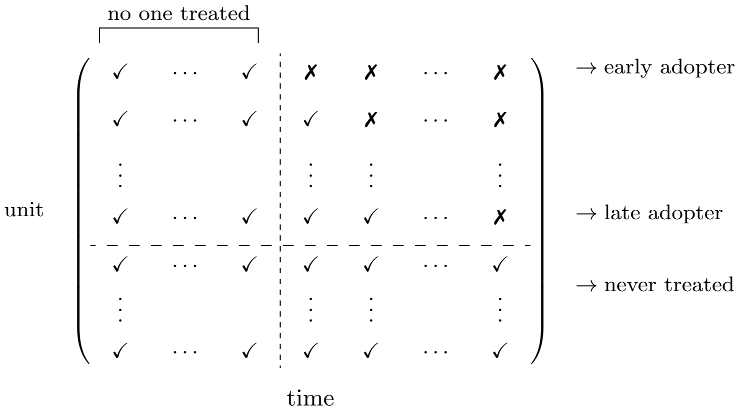


not treated : ✓; treated : ✗

Quantity of interest: (individual) treatment effect $Y_{1T}(1) - Y_{1T}(0)$ (random!)

Data Structure: Staggered Adoption

Adoption time: T_i Outcome: $Y_{it} = Y_{it}(T_i)\mathbb{1}(t \geq T_i) + Y_{it}(\infty)\mathbb{1}(t < T_i)$



not treated : ✓; treated : ✗

Quantity of interest: much more flexible

Data Structure: Staggered Adoption

Adoption time: T_i Outcome: $Y_{it} = Y_{it}(T_i)\mathbb{1}(t \geq T_i) + Y_{it}(\infty)\mathbb{1}(t < T_i)$

Many quantities of interest. For example,

- Individual treatment effect after k periods:

$$Y_{i(T_i+k)}(T_i) - Y_{i(T_i+k)}(\infty)$$

- Average treatment effect on units treated at time s after k periods:

$$\frac{1}{\#\{j : T_j = s\}} \sum_{i:T_i=s} \left(Y_{i(s+k)}(s) - Y_{i(s+k)}(\infty)\right)$$

  - Apply SC to each treated unit or an "aggregate" unit

- Average treatment effect on the treated (size N_1) after k periods:

$$\frac{1}{N_1} \sum_{i=1}^{N_1} \left(Y_{i(T_i+k)}(T_i) - Y_{i(T_i+k)}(\infty)\right)$$

Synthetic Control: Basics

In the canonical case,

$$Y_{it} = \begin{cases} Y_{it}(0) & \text{if } i = 2, \ldots N+1 \\ Y_{it}(0) & \text{if } i = 1 \text{ and } t \in \{1, 2, \ldots, T_0\} \\ Y_{it}(1) & \text{if } i = 1 \text{ and } t \in \{T_0 + 1, \ldots, T_0 + T_1\}. \end{cases}$$

Treatment effect on the treated (random!)

$$\tau_T = Y_{1T}(1) - Y_{1T}(0), \quad \text{for} \quad T > T_0$$

Find $\{w_i\}$

$$\sum_{i=2}^{N+1} w_i Y_{it}(0) \approx Y_{1t}(0), \quad \text{for} \quad t = 1, \cdots, T_0,$$

Hopefully,

$$\sum_{i=2}^{N+1} w_i Y_{iT}(0) \approx Y_{1T}(0), \quad \text{for} \quad T > T_0.$$

- Intuition: stable cross-sectional relation over time

Synthetic Control: Basics

In the canonical case,

$$Y_{it} = \begin{cases} Y_{it}(0) & \text{if } i = 2, \ldots N+1 \\ Y_{it}(0) & \text{if } i = 1 \text{ and } t \in \{1, 2, \ldots, T_0\} \\ Y_{it}(1) & \text{if } i = 1 \text{ and } t \in \{T_0+1, \ldots, T_0+T_1\}. \end{cases}$$

Treatment effect on the treated (random!)

$$\tau_T = Y_{1T}(1) - Y_{1T}(0), \quad \text{for} \quad T > T_0$$

Find $\{w_i\}$

$$\sum_{i=2}^{N+1} w_i Y_{it}(0) \approx Y_{1t}(0), \quad \text{for} \quad t = 1, \cdots, T_0,$$

Canonical SC

$$\widehat{\boldsymbol{w}} = \arg\min_{\boldsymbol{w} \in \mathcal{W}} \sum_{t=1}^{T_0} \left(Y_{1t}(0) - \boldsymbol{x}_t' \boldsymbol{w} \right)^2,$$

- $\mathcal{W} = \{\boldsymbol{w} \in \mathbb{R}^N : \sum_{i=2}^{N+1} w_i = 1, w_i \geq 0, \ \forall i \geq 2\}$ $\qquad \boldsymbol{x}_t = (Y_{2t}(0), \cdots, Y_{N+1,t}(0))'$

Prediction Intervals

Treatment effect on the treated

$$\tau_T = Y_{1T}(1) - Y_{1T}(0), \qquad \widehat{\tau}_T = Y_{1T}(1) - \boldsymbol{x}_T'\widehat{\boldsymbol{w}}, \qquad T > T_0$$

Prediction interval (PI) for *random* τ_T

$$\mathbb{P}\Big\{\mathbb{P}\Big[\tau_T \in \mathcal{I} \,\Big|\, \mathscr{H}\Big] \geq 1-\alpha\Big\} \geq 1-\pi$$

- $\mathcal{I}$: prediction interval
- $\mathscr{H}$: conditioning σ-algebra
  - unconditional if $\mathscr{H}$ is trivial
  - This paper: $\{\boldsymbol{x}_t : 1 \leq t \leq T\}$
- $(1-\alpha)$: conditional coverage prob., e.g., 95%
- π: failure over $\mathscr{H}$

Two Sources of Uncertainty

Treatment effect on the treated

$$\tau_T = Y_{1T}(1) - Y_{1T}(0), \qquad \widehat{\tau}_T = Y_{1T}(1) - \boldsymbol{x}_T' \widehat{\boldsymbol{w}}$$

For a pseudo-true value $\boldsymbol{w}_0$ ("estimand" of $\widehat{\boldsymbol{w}}$) ▸ Def.

$$Y_{1T}(0) = \boldsymbol{x}_T' \boldsymbol{w}_0 + u_T$$

A simple decomposition

$$\begin{aligned} \widehat{\tau}_T - \tau_T &= \quad Y_{1T}(0) \quad - \boldsymbol{x}_T' \widehat{\boldsymbol{w}} \\ &= (\boldsymbol{x}_T' \boldsymbol{w}_0 + u_T) - \boldsymbol{x}_T' \widehat{\boldsymbol{w}} \\ &= \quad u_T \quad - \boldsymbol{x}_T' (\widehat{\boldsymbol{w}} - \boldsymbol{w}_0) \end{aligned}$$

- In-sample error: $\boldsymbol{x}_T' (\widehat{\boldsymbol{w}} - \boldsymbol{w}_0)$
- Out-of-sample error: u_T
- Non-asymptotically, both are important

Prediction Intervals: Basic Construction

$$\widehat{\tau}_T - \tau_T = u_T - \boldsymbol{x}_T'(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0)$$

- **In-Sample Error**: with prob. $\geq 1 - \pi_1$ (over $\mathscr{H}$)

$$\mathbb{P}\Big[M_{1,\mathtt{L}} \leq \boldsymbol{x}_T'(\boldsymbol{w}_0 - \widehat{\boldsymbol{w}}) \leq M_{1,\mathtt{U}} \;\Big|\; \mathscr{H}\Big] \geq 1 - \alpha_1$$

- **Out-of-Sample Error**: with prob. $\geq 1 - \pi_2$ (over $\mathscr{H}$)

$$\mathbb{P}\Big[M_{2,\mathtt{L}} \leq u_T \leq M_{2,\mathtt{U}} \;\Big|\; \mathscr{H}\Big] \geq 1 - \alpha_2$$

Prediction Interval for τ_T: with prob. $\geq 1 - \pi_1 - \pi_2$ (over $\mathscr{H}$)

$$\mathbb{P}\Big[M_{1,\mathtt{L}} + M_{2,\mathtt{L}} \leq \widehat{\tau}_T - \tau_T \leq M_{1,\mathtt{U}} + M_{2,\mathtt{U}} \Big| \mathscr{H}\Big] \geq 1 - \alpha_1 - \alpha_2$$

- Conservative, but offer non-asymptotic probability guarantee

In-Sample Error: Stationary Case

$$\sqrt{T_0}(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0) = \underset{\boldsymbol{\delta} \in \sqrt{T_0}(\mathcal{W} - \boldsymbol{w}_0)}{\arg\min} \; \ell(\boldsymbol{\delta}), \quad \ell(\boldsymbol{\delta}) = \boldsymbol{\delta}' \underbrace{\left(\frac{1}{T_0} \sum_{t=1}^{T_0} \boldsymbol{x}_t \boldsymbol{x}_t' \right)}_{\widehat{\boldsymbol{Q}}} \boldsymbol{\delta} - 2 \underbrace{\left(\frac{1}{\sqrt{T_0}} \sum_{t=1}^{T_0} \boldsymbol{x}_t' u_t \right)}_{\widehat{\boldsymbol{\gamma}}} \boldsymbol{\delta}$$

- Assume $\mathcal{W}$ is convex
- Possibly **misspecified**: $\boldsymbol{\gamma} := \mathbb{E}[\widehat{\boldsymbol{\gamma}} | \mathscr{H}] \neq 0$
- By optimality of $\widehat{\boldsymbol{w}}$ and $\boldsymbol{w}_0$,

$$\sqrt{T_0}(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0) \in \left\{ \boldsymbol{\delta} \in \Delta : \boldsymbol{\delta}' \widehat{\boldsymbol{Q}} \boldsymbol{\delta} - 2(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})' \boldsymbol{\delta} \leq 0 \right\}, \quad \Delta = \sqrt{T_0}(\mathcal{W} - \boldsymbol{w}_0)$$

  - $\widehat{\boldsymbol{Q}}$ is fixed conditional on $\mathscr{H}$
  - Approximate $\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$

In-Sample Error: Distributional Approximation

Goal: bound $\boldsymbol{x}_T'(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0)$, e.g.

$$\sqrt{T_0}\boldsymbol{x}_T'(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0) \leq \sup_{\boldsymbol{\delta} \in \mathcal{M}_{\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}}} \boldsymbol{x}_T'\boldsymbol{\delta}$$

$$\mathcal{M}_{\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}} = \left\{\boldsymbol{\delta} \in \Delta : \boldsymbol{\delta}'\widehat{\boldsymbol{Q}}\boldsymbol{\delta} - 2(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})'\boldsymbol{\delta} \leq 0\right\}$$

Reduce to distributional approximation of $\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$: for any κ,

$$\sup_{\boldsymbol{\delta} \in \mathcal{M}_{\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}}} \boldsymbol{x}_T'\boldsymbol{\delta} \leq \kappa \quad \Leftrightarrow \quad \widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma} \in \mathcal{A}_\kappa$$

- Normal approximation: ▸ detail

$$\begin{aligned} \mathbb{P}(\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma} \in \mathcal{A}_\kappa) &\approx \mathbb{P}(\boldsymbol{G} \in \mathcal{A}_\kappa), \quad \boldsymbol{G} \sim \mathsf{N}(0, \mathbb{V}[\widehat{\boldsymbol{\gamma}}|\mathscr{H}]) \\ &\approx \mathbb{P}(\widehat{\boldsymbol{G}} \in \mathcal{A}_\kappa), \quad \widehat{\boldsymbol{G}} \sim \mathsf{N}(0, \widehat{\mathbb{V}}[\widehat{\boldsymbol{\gamma}}|\mathscr{H}]) \end{aligned}$$

- Bounds can be obtained by simulation
  - A constraint set $\Delta^\star$ in simulation: locally equivalent to $\Delta = \sqrt{T_0}(\mathcal{W} - \boldsymbol{w}_0)$

Out-of-Sample Error

Three approaches:

- Concentration inequalities, e.g. subgaussian

$$\mathbb{P}\Big(|u_T - \mathbb{E}[u_T|\mathscr{H}]| \geq \varpi_u \Big| \mathscr{H}\Big) \leq 2\exp\left(-\frac{\varpi_u^2}{2\sigma_{\mathscr{H}}^2}\right)$$

- Location-scale model

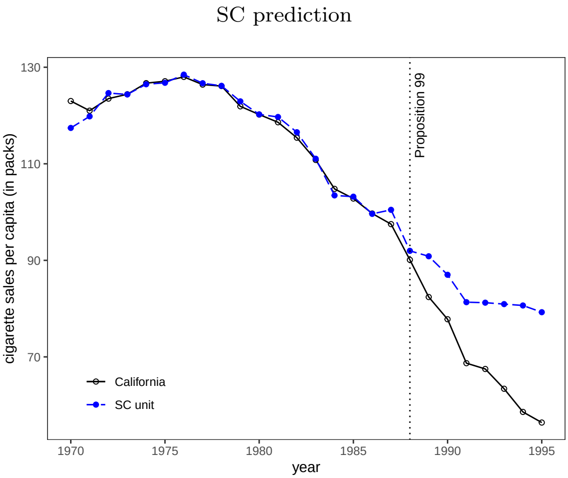
$$u_t = \mathbb{E}[u_t|\mathscr{H}] + (\mathbb{V}[u_t|\mathscr{H}])^{1/2} e_t, \quad \{e_t\} \perp\!\!\!\perp \mathscr{H}$$

- Quantile regression: model conditional quantiles of u_t given $\mathscr{H}$

Sensitivity analysis

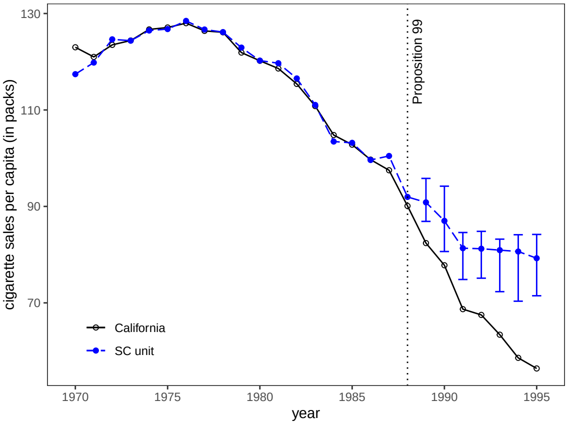
- To cancel out the effect, how large u_T needs to be?

Example: Proposition 99 (California)



Example: Proposition 99 (California)

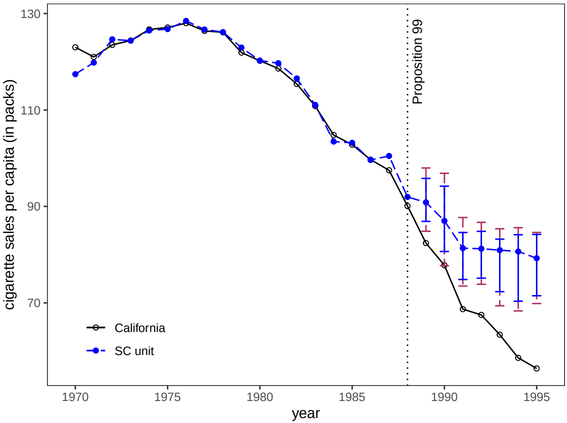
SC prediction with PI, in-sample error



Note: PI has $\geq 95\%$ coverage.

Example: Proposition 99 (California)

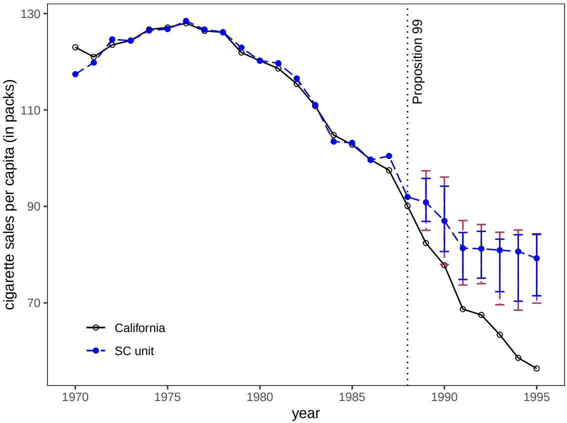
SC prediction with PI for $Y_{1T}(0)$, concentration-based



Note: PI has $\geq 90\%$ coverage.

Example: Proposition 99 (California)

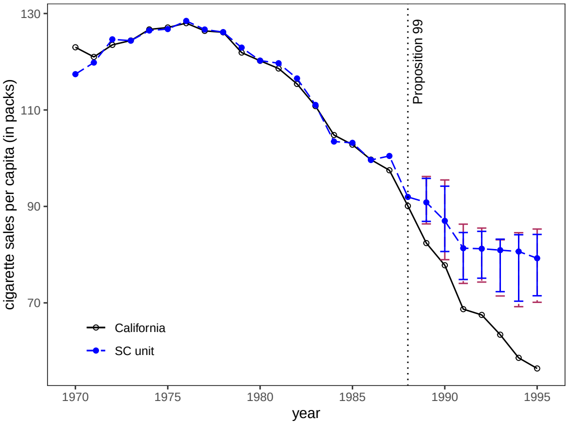
SC prediction with PI for $Y_{1T}(0)$, local-scale model



Note: PI has $\geq 90\%$ coverage.

Example: Proposition 99 (California)

SC prediction with PI for $Y_{1T}(0)$, quantile-based



Note: PI has $\geq 90\%$ coverage.

Extension: Multiple Treated Units, Staggered Adoption

- $\mathfrak{J}$ treated units
- Staggered adoption: $T_1 \leq \cdots \leq T_{N_1}$, $T_0 := T_1 - 1$
- M features (not just pre-treatment outcomes)
- General convex constraint set $\mathcal{W}$ (possibly nonlinear)
- Weighting matrix $\boldsymbol{V}$

$$\underbrace{\boldsymbol{A}_j}_{\substack{\text{treated} \\ T_0 \cdot M \times 1}} \approx \underbrace{\boldsymbol{B}_j}_{\substack{\text{untreated} \\ T_0 \cdot M \times N}} \boldsymbol{w}_j, \qquad \text{for} \quad j = 1, \cdots, \mathfrak{J},$$

$$\underbrace{\boldsymbol{A}}_{T_0 \cdot M \cdot \mathfrak{J} \times 1} = \begin{bmatrix} \boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{A}_{\mathfrak{J}} \end{bmatrix}, \qquad \underbrace{\boldsymbol{B}}_{T_0 \cdot M \cdot \mathfrak{J} \times N \cdot \mathfrak{J}} = \begin{bmatrix} \boldsymbol{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{B}_{\mathfrak{J}} \end{bmatrix}$$

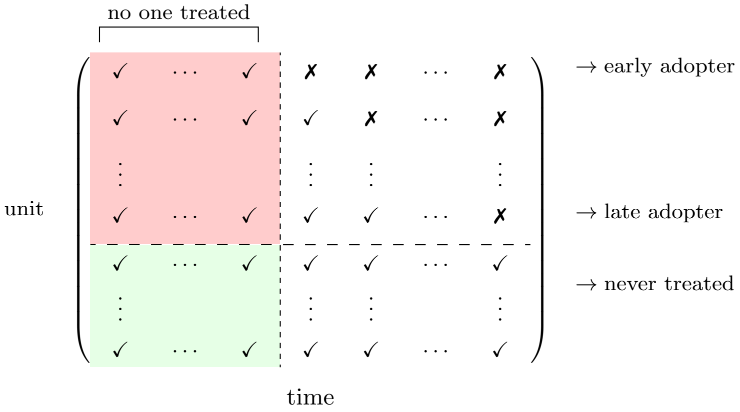
$$\widehat{\boldsymbol{w}} = (\widehat{\boldsymbol{w}}_1', \cdots, \widehat{\boldsymbol{w}}_{\mathfrak{J}}')' = \underset{\boldsymbol{w} \in \mathcal{W}}{\arg\min}\, (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{w})' \boldsymbol{V} (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{w})$$

Similar optimization bounds can be constructed

Extension: Multiple Treated Units, Staggered Adoption

For example,

- Fix the sample used to construct weights
- $\boldsymbol{A}$ matrix: use the "red" portion
- $\boldsymbol{B}$ matrix: use the "green" portion



Extension: Simultaneous Prediction Intervals

- The previous PIs have coverage for *each* post-treatment period
- Can be generalized to have **simultaneous** coverage for multiple periods
- For example, in the canonical case, treatment effect on "unit 1" at time t is

$$\tau_t = Y_{1t}(1) - Y_{1t}(0)$$
$$\hat{\tau}_t - \tau_t = u_t - \boldsymbol{x}_t'(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0)$$

Simultaneous PI:

$$\mathbb{P}\Big\{\mathbb{P}\big[\tau_t \in \mathcal{I}, \text{ for all } T_0 + 1 \leq t \leq T_0 + L \,\big|\, \mathscr{H}\big] \geq 1 - \alpha\Big\} \geq 1 - \pi.$$

- In-sample error: easy to accommodate in the optimization
  - Take sup/inf over more evaluation vectors $\{\boldsymbol{x}_t\}$
- Out-of-sample error:
  - Maximal inequality for $\{u_t\}$

Conclusion

- Prediction Intervals for general SC methods
- Conditional validity
- Non-asymptotic probability guarantee
- Two sources of uncertainty
- Staggered adoption allowed

Thanks!

In-Sample Error: Plug-in Approximation

$$\mathbb{P}(\sqrt{T_0}\boldsymbol{x}_T'(\widehat{\boldsymbol{w}}-\boldsymbol{w}_0)\leq\kappa)\geq\mathbb{P}(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}\in\mathcal{A}_\kappa)\approx\mathbb{P}(\boldsymbol{G}\in\mathcal{A}_\kappa),\quad \boldsymbol{G}\sim\mathsf{N}(0,\mathbb{V}[\widehat{\boldsymbol{\gamma}}|\mathscr{H}])$$
$$\approx\mathbb{P}(\widehat{\boldsymbol{G}}\in\mathcal{A}_\kappa),\quad \widehat{\boldsymbol{G}}\sim\mathsf{N}(0,\widehat{\mathbb{V}}[\widehat{\boldsymbol{\gamma}}|\mathscr{H}])$$

Basic approximation strategy:

$$\sup\left\{\boldsymbol{x}_T'\boldsymbol{\delta}:\boldsymbol{\delta}\in\Delta,\ \ \boldsymbol{\delta}'\widehat{\boldsymbol{Q}}\boldsymbol{\delta}-2(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma})'\boldsymbol{\delta}\leq 0\right\}$$
$$\Downarrow$$
$$\sup\left\{\boldsymbol{x}_T'\boldsymbol{\delta}:\boldsymbol{\delta}\in\Delta,\ \ \boldsymbol{\delta}'\widehat{\boldsymbol{Q}}\boldsymbol{\delta}-2\boldsymbol{G}'\boldsymbol{\delta}\leq 0\right\}$$
$$\Downarrow$$
$$\sup\left\{\boldsymbol{x}_T'\boldsymbol{\delta}:\boldsymbol{\delta}\in\Delta^\star,\ \boldsymbol{\delta}'\widehat{\boldsymbol{Q}}\boldsymbol{\delta}-2\widehat{\boldsymbol{G}}'\boldsymbol{\delta}\leq 0\right\}$$

- $\Delta^\star$: locally equivalent to $\Delta=\sqrt{T_0}(\mathcal{W}-\boldsymbol{w}_0)$
- Thresholding

What is $\boldsymbol{w}_0$?

Definition of $\boldsymbol{w}_0$ relies on

- Constraint $\mathcal{W}$
- Assumptions on $\{Y_{1t}(0), \boldsymbol{x}_t\}$
- Conditioning set $\mathscr{H}$ ($\boldsymbol{w}_0$ may be random)

Examples

- $(Y_{1t}(0), \boldsymbol{x}_t)$ stationary

$$\boldsymbol{w}_0 = \arg\min_{\boldsymbol{w}\in\mathcal{W}} \mathbb{E}\left[\frac{1}{T_0}\sum_{t=1}^{T_0}\left(Y_{1t}(0) - \boldsymbol{x}_t'\boldsymbol{w}\right)^2 \Big| \mathscr{H}\right]$$

  - (Constrained) best linear prediction

- $Y_{1t}(0), \boldsymbol{x}_t \sim I(1)$ (integrated process)
  - Cointegration: $Y_{1t}(0) - \boldsymbol{x}_t'\boldsymbol{w}_0 \sim I(0)$
  - $(1, -\boldsymbol{w}_0)$: cointegrating vector (unique given $\boldsymbol{w}_0 \in \mathcal{W}$)

back

In-Sample Error: Non-Stationary Case

$$T_0(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0) = \underset{\boldsymbol{\delta} \in T_0(\mathcal{W} - \boldsymbol{w}_0)}{\arg\min} \ \ell(\boldsymbol{\delta}), \quad \ell(\boldsymbol{\delta}) = \boldsymbol{\delta}' \underbrace{\Big(\frac{1}{T_0^2} \sum_{t=1}^{T_0} \boldsymbol{x}_t \boldsymbol{x}_t' \Big)}_{\widehat{\boldsymbol{Q}}} \boldsymbol{\delta} - 2 \underbrace{\Big(\frac{1}{T_0} \sum_{t=1}^{T_0} \boldsymbol{x}_t' u_t \Big)}_{\widehat{\boldsymbol{\gamma}}} \boldsymbol{\delta}$$

Cointegrated system, e.g.,

$$Y_{1t}(0) = \boldsymbol{x}_t' \boldsymbol{w}_0 + u_t, \quad \boldsymbol{x}_t = \boldsymbol{x}_{t-1} + \boldsymbol{\epsilon}_t, \quad (u_t, \boldsymbol{\epsilon}_t) \sim \text{i.i.d } I(0)$$

Again, bound $T_0(\widehat{\boldsymbol{w}} - \boldsymbol{w}_0)$ by simulating

$$\sup \Big\{ \boldsymbol{x}_T' \boldsymbol{\delta} : \boldsymbol{\delta} \in \Delta^\star, \ \boldsymbol{\delta}' \widehat{\boldsymbol{Q}} \boldsymbol{\delta} - 2 \widehat{\boldsymbol{G}}' \boldsymbol{\delta} \leq 0 \Big\}, \quad \widehat{\boldsymbol{G}} \sim \mathsf{N}(0, \widehat{\mathbb{V}}[\widehat{\boldsymbol{\gamma}} | \mathscr{H}])$$

- $\widehat{\boldsymbol{Q}}$ fixed conditional on $\mathscr{H}$
- Probably $\mathbb{E}[u_t | \mathscr{H}] \neq 0$
- Non-stationarity affects the analysis of $\widehat{\boldsymbol{Q}}$ and $\mathbb{V}[\widehat{\boldsymbol{\gamma}} | \mathscr{H}]$